

# A proof of the Breuil-Schneider conjecture in the indecomposable case

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## Abstract

This paper contains a proof of a conjecture of Breuil and Schneider, on the existence of an invariant norm on any locally algebraic representation of  $\mathrm{GL}(n)$ , with integral central character, whose smooth part is given by a generalized Steinberg representation. In fact, we prove the analogue for any connected reductive group  $G$ . This is done by passing to a global setting, using the trace formula for an  $\mathbb{R}$ -anisotropic model of  $G$ . The ultimate norm comes from classical  $p$ -adic modular forms. <sup>1 2</sup>

## 1 Introduction

The  $p$ -adic Langlands program is still in its infancy. For a  $p$ -adic field  $F$ , one anticipates a correspondence between certain Galois representations  $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/F) \rightarrow \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$  and certain representations  $\hat{\pi}$  of  $\mathrm{GL}_n(F)$  on  $p$ -adic Banach spaces. See Breuil's survey [Br] from the ICM 2010. This correspondence should somehow be compatible with reduction mod  $p$ , cohomology, and  $p$ -adic families. This is a (big) theorem for  $\mathrm{GL}_2(\mathbb{Q}_p)$ , due to the work of many people (Berger, Breuil, Colmez, Paskunas, and others). However, beyond this example next to nothing is known. Even  $\mathrm{GL}_2(F)$ , for fields  $F \neq \mathbb{Q}_p$ , seems surprisingly hard to deal with. Let us return to  $\mathrm{GL}_2(\mathbb{Q}_p)$  for a moment, and give more details: We start off with a potentially semistable Galois representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \rightarrow \mathrm{GL}(V) \simeq \mathrm{GL}_2(E),$$

with coefficients in a finite extension  $E/\mathbb{Q}_p$ . We assume  $\rho$  is *regular*. That is, it has distinct Hodge-Tate weights  $w_1 < w_2$ . By a standard recipe of Fontaine, to be recalled below, one associates a Weil-Deligne representation  $\mathrm{WD}(\rho)$ . By the classical local Langlands correspondence, its Frobenius-semisimplification  $\mathrm{WD}(\rho)^{F-ss}$  corresponds to an irreducible smooth representation  $\pi'$  of  $\mathrm{GL}_2(\mathbb{Q}_p)$  over  $E$ . We let  $\pi = \pi' \otimes |\det|^{-1/2}$  if  $\pi'$  is generic (that is, infinite-dimensional). If  $\pi'$  is non-generic, we replace it by  $\pi = \pi'' \otimes |\det|^{-1/2}$ , where  $\pi''$  is a certain parabolically induced representation with  $\pi'$  as its unique irreducible quotient. This is the *generic* local Langlands correspondence. Note that  $\pi$  may be reducible. Now, one attaches to  $\rho$  an admissible unitary Banach space representation  $B(\rho)$  of  $\mathrm{GL}_2(\mathbb{Q}_p)$  over  $E$  satisfying a list of desiderata [Br, p. 8]. Most

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important for us, is that  $B(\rho)$  is the completion, relative to a suitable invariant norm, of the locally algebraic representation (at least when  $\rho$  is irreducible):

$$B(\rho)^{alg} = \det^{w_1} \otimes_E \mathrm{Sym}^{w_1 - w_2 - 1}(E^2) \otimes_E \pi.$$

Moreover,  $B(\cdot)$  is compatible with the mod  $p$  local Langlands correspondence.

The Breuil-Schneider conjecture mimics some of this for  $\mathrm{GL}_n(F)$ . Again, let

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/F) \rightarrow \mathrm{GL}(V) \simeq \mathrm{GL}_n(E)$$

be a potentially semistable Galois representation. With  $\rho$ , we associate a Weil-Deligne representation  $\mathrm{WD}(\rho)$  and a multiset of integers  $\mathrm{HT}(\rho)$  as follows: Pick a finite Galois extension  $F'/F$  such that  $\rho|_{\mathrm{Gal}(\bar{\mathbb{Q}}_p/F')}$  is semistable. Then

$$D = (B_{st} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\bar{\mathbb{Q}}_p/F')}$$

is a free  $F'_0 \otimes_{\mathbb{Q}_p} E$ -module of rank  $n$ , where  $F'_0$  is the maximal unramified subfield of  $F'$ . The module  $D$  comes equipped with a Frobenius  $\phi$ , a monodromy operator  $N$ , such that  $N\phi = p\phi N$ , and a commuting action of  $\mathrm{Gal}(F'/F)$ . Moreover, there is an admissible filtration of  $D_{F'}$  by  $\mathrm{Gal}(F'/F)$ -invariant  $F' \otimes_{\mathbb{Q}_p} E$ -submodules, which allows to recover  $\rho$ . Observe that one has a factorization,

$$D_{F'} \simeq \prod_{\sigma: F \rightarrow E} D_{F', \sigma}, \quad D_{F', \sigma} = D_{F'} \otimes_{F' \otimes_{\mathbb{Q}_p} E} (F' \otimes_{F, \sigma} E).$$

Hence, for each  $\sigma$ , we are given a filtration  $\mathrm{Fil}^i(D_{F', \sigma})$  by  $\mathrm{Gal}(F'/F)$ -invariant free  $F' \otimes_{F, \sigma} E$ -submodules. Admissibility means, intuitively, that the Hodge polygon lies beneath the Newton polygon. More formally, one introduces numbers  $t_N(D)$  and  $t_H(D_{F'})$  as in [BS, p. 15]. The former is given purely in terms of  $\phi$ , the latter in terms of the filtration. One requires that  $t_H(D_{F'}) = t_N(D)$ , and that  $t_H(D'_{F'}) \leq t_N(D')$  for any subobject  $D' \subset D$  (with the induced filtration).

*Hodge-Tate numbers:* For every embedding  $\sigma : F \rightarrow E$ , the  $n$ -element multiset  $\mathrm{HT}_\sigma(\rho)$  contains  $i \in \mathbb{Z}$  with multiplicity  $\mathrm{rk}_{(F' \otimes_{F, \sigma} E)} \mathrm{gr}^i(D_{F', \sigma})$ . We label these,

$$\mathrm{gr}^i(D_{F', \sigma}) \neq 0 \Leftrightarrow i \in \mathrm{HT}_\sigma(\rho) = \{i_{1, \sigma} \leq \dots \leq i_{n, \sigma}\}.$$

We say  $\rho$  is regular (at  $\sigma$ ) if all the Hodge-Tate numbers  $i_{j, \sigma}$  are distinct.

*Weil-Deligne representation:* Forgetting the filtration, the  $(\phi, N)$ -module  $D$  gives rise to  $\mathrm{WD}(\rho)$  as follows. Choose an embedding  $F'_0 \hookrightarrow E$  and consider  $D_E = D \otimes_{F'_0 \otimes_{\mathbb{Q}_p} E} E$  with the inherited monodromy operator  $N$ , and  $W_F$ -action

$$r(w) = \phi^{-d(w)} \circ \bar{w}, \quad w \in W_F.$$

(Here  $d(w)$  is the power of arithmetic Frobenius induced by  $w$ , its image in  $\mathrm{Gal}(F'/F)$  is  $\bar{w}$ , and  $\phi$  is the semilinear Frobenius on  $B_{st}$ .) Note that  $r|_{W_{F'}}$  is unramified. This defines  $\mathrm{WD}(\rho) = (r, N, D_E)$ , a Weil-Deligne representation.

Conversely, suppose we are given a Frobenius-semisimple Weil-Deligne representation  $(r, N, D_E)$  of  $W_F$  over  $E$ , unramified when restricted to  $W_{F'}$ , and for each  $\sigma : F \rightarrow E$  a set of  $n$  distinct integers,

$$i_{1, \sigma} < \dots < i_{n, \sigma}.$$

When does these data arise from a potentially semistable  $\rho$ ? By [BS, p. 14] we know  $(r, N, D_E)$  corresponds to a  $(\phi, N) \times \text{Gal}(F'/F)$ -module  $D$ . What we are asking for, is an admissible filtration  $\text{Fil}^i(D_{F', \sigma})$  such that

$$\text{gr}^i(D_{F', \sigma}) \neq 0 \Leftrightarrow i \in \{i_{1, \sigma} < \cdots < i_{n, \sigma}\}.$$

The Breuil-Schneider conjecture asserts this is the case precisely when some locally algebraic representation  $\xi \otimes_E \pi$  (constructed from the given data) carries an invariant norm. That is, a non-archimedean norm  $\|\cdot\|$  such that  $\text{GL}_n(F)$  acts unitarily.

*The algebraic representation  $\xi$ :* This is constructed out of the tuples  $i_{j, \sigma}$ . Let

$$a_{j, \sigma} = -i_{n+1-j, \sigma} - (j-1), \quad a_{1, \sigma} \leq \cdots \leq a_{n, \sigma}.$$

That is, write  $i_{j, \sigma}$  in the opposite order, change signs, subtract  $(0, 1, \dots, n-1)$ . The sequence  $a_{j, \sigma}$  is identified with a dominant weight for  $\text{GL}_n$ , relative to the lower triangular Borel. We let  $\xi_\sigma$  be the corresponding irreducible algebraic representation of  $\text{GL}_n$ , and  $\xi = \otimes \xi_\sigma$ , viewed as an irreducible algebraic representation of the restriction of scalars  $\text{Res}_{F/\mathbb{Q}_p} \text{GL}_n$ , over  $E$ .

*The smooth representation  $\pi$ :* This is constructed out of  $(r, N, D_E)$  via a modified local Langlands correspondence. Let  $\pi^\circ$  be the smooth irreducible representation of  $\text{GL}_n(F)$  (over  $\mathbb{Q}_p$ ) associated with  $(r, N, D_E)$  by the usual unitary local Langlands correspondence (after fixing a square root of  $q = \#\mathbb{F}_F$ ),

$$(r, N, D_E) \simeq \text{rec}(\pi^\circ \otimes |\det|^{(1-n)/2}).$$

The twist  $\pi^\circ(\frac{1-n}{2})$  does not depend on the choice of  $q^{\frac{1}{2}}$ , and can be defined over  $E$ . By the Langlands classification (see [Ku] for a useful survey),  $\pi^\circ$  is the unique irreducible quotient of a parabolically induced representation,

$$\text{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r)) \twoheadrightarrow Q(\Delta_1, \dots, \Delta_r) \simeq \pi^\circ.$$

Here the induction is normalized. The  $Q(\Delta_i)$  are generalized Steinberg representations, built from segments of supercuspidals,  $\Delta_i$ , ordered in a suitable way. We define

$$\pi = \text{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r)) \otimes |\det|^{(1-n)/2}.$$

By [BS, p. 16], this  $\pi$  can be defined over  $E$ . Note that  $\pi$  may be reducible, and it admits  $\pi^\circ(\frac{1-n}{2})$  as its unique irreducible quotient. Moreover,  $\pi \simeq \pi^\circ(\frac{1-n}{2})$  exactly when the representation  $\pi^\circ$  is generic. Also,  $\pi$  is always generic [JS]. This is the so-called generic local Langlands correspondence for  $\text{GL}_n$ .

We are now in a position to state the conjecture, announced in [BS] and [Br].

**The Breuil-Schneider conjecture.** *Fix data  $(r, N, D_E)$  and  $i_{j, \sigma}$  as above, and let  $\pi$  and  $\xi$  be the representations constructed therefrom. Then the following two conditions are equivalent,*

- (1) *The data arises from a potentially semistable Galois representation.*
- (2) *The representation  $\xi \otimes_E \pi$  admits a  $\text{GL}_n(F)$ -invariant norm  $\|\cdot\|$ .*

The implication (2)  $\Rightarrow$  (1) is completely known. A few cases were worked out in [BS], and Hu proved it in general in [Hu]. In fact, Hu proves a lot more. He shows that (1) is *equivalent* to what he refers to as the Emerton condition, which is a purely group theoretic statement: With  $V$  denoting the space  $\xi \otimes_E \pi$ ,

$$(3) \quad V^{N_0, Z_M^+ = x} \neq 0 \Rightarrow |\delta_P^{-1}(z)\chi(z)| \leq 1,$$

for all  $z \in Z_M^+$ . The implication (2)  $\Rightarrow$  (3) is an easy exercise.

We are concerned with the converse, (1)  $\Rightarrow$  (2). Our main result is:

**Theorem A.** *The conjecture holds when  $(r, N, D_E)$  is indecomposable.*

Recall that indecomposable Weil-Deligne representations are precisely those obtained as follows: Starting with an irreducible representation  $\tilde{r} : W_F \rightarrow \mathrm{GL}(\tilde{D})$ , with open kernel, and a positive integer  $s \in \mathbb{Z}_{>0}$ , let

$$D = \tilde{D}^{\oplus s}, \quad r = \tilde{r} \oplus \tilde{r}(1) \oplus \cdots \oplus \tilde{r}(s-1), \quad N : \tilde{r}(i-1) \xrightarrow{\sim} \tilde{r}(i).$$

Here  $\tilde{r}(i)$  denotes twisting  $\tilde{r}$  by the  $i$ th power of  $|\cdot|$ , the absolute value on  $W_F$ , transferred from  $F^*$  via the reciprocity map. Under the (classical) local Langlands correspondence,  $\tilde{D}$  corresponds to a supercuspidal  $\tau$ , and  $D$  corresponds to the generalized Steinberg representation  $Q(\Delta)$ , where  $\Delta$  is the segment

$$\Delta = \tau \otimes \tau(1) \otimes \cdots \otimes \tau(s-1).$$

The Jacquet modules of  $Q(\Delta)$  can be made explicit, see Lemma 3.1 in [Hu], for example. They are irreducible if nonzero. From that, it is easy to see that condition (3) just amounts to saying  $\xi \otimes_E \pi$  has integral central character. In fact, this was already observed in Proposition 5.3 in [BS], where they also state the resulting conjecture explicitly (as Conjecture 5.5), which is what we prove. Our methods work for any connected reductive group  $G$  defined over  $\mathbb{Q}_p$ .

**Theorem B.** *Let  $G$  be a connected reductive group over  $\mathbb{Q}_p$ . Let  $\xi$  be any irreducible algebraic representation of  $G_{\mathbb{Q}_p}$ , and  $\pi$  be any essentially discrete series representation of  $G$ . Then  $\xi \otimes \pi$  admits a  $G$ -invariant norm if and only if its central character is integral.*

Taking  $G = \mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}(n)$ , yields Conjecture 5.5 in [BS]. Indeed, the generalized Steinberg representations coincide with the essentially discrete series representations, for  $\mathrm{GL}(n)$ . This Theorem, and its proof, is purely group-theoretical. There is no mention of Galois representations, and much of the previous discussion is meant to be motivation only.

The proof of Theorem B (which implies Theorem A) is by passing to a global setting, and making use of algebraic modular forms. By some sort of averaging over finite (cohomology) groups, we first reduce to the case where  $G$  is simple and simply connected, in which case the condition on the central character is vacuous. For such  $G$ , a result of Borel and Harder allows us to find a global model  $G/\mathbb{Q}$  such that  $G(\mathbb{R})$  is compact. If  $\pi$  is a discrete series, a trace formula argument (due to Clozel in greater generality) shows that  $\xi \otimes \pi$  admits an automorphic extension. Fixing an isomorphism  $\iota : \mathbb{C} \rightarrow \bar{\mathbb{Q}}_p$ , we infer that  $\pi^K$  sits as a submodule of  $\mathcal{A}_{G,\xi}^K$ , a space of classical  $p$ -adic modular forms. Therefore,  $\xi \otimes \pi$  contributes to the direct limit of all  $\xi \otimes \mathcal{A}_{G,\xi}^K$ , which in turn embeds in

$\mathcal{C}_G$ , the space of all continuous functions  $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \rightarrow \bar{\mathbb{Q}}_p$ . This latter space carries a supremum-norm, which is obviously invariant under  $G(\mathbb{A}_f)$ .

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## 2 Modular forms on definite reductive groups

### 2.1 The complex case

#### 2.1.1 Notation

For now, we will study automorphic forms on an arbitrary connected reductive group  $G$  over  $\mathbb{Q}$  such that  $G^{\text{der}}(\mathbb{R})$  is compact. Here  $G^{\text{der}}$  is the derived subgroup, which is then necessarily an  $\mathbb{R}$ -anisotropic semisimple group. As is standard,  $A_G$  denotes the maximal  $\mathbb{Q}$ -split central torus in  $G$ , and we choose any central torus  $Z_G$  (over  $\mathbb{Q}$ ) containing  $A_G$ . We will often take it to be the whole identity component of the center.  $K_\infty$  is the maximal compact subgroup of  $G(\mathbb{R})$ , which is unique, and possibly bigger than  $G^{\text{der}}(\mathbb{R})$ .

#### 2.1.2 Classical automorphic forms

Let  $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$  be the ring of rational adeles. Inside  $G(\mathbb{A})$ , we introduce the normal subgroup  $G(\mathbb{A})^1$  cut out by all  $|\chi|$ , where  $\chi$  ranges over the  $\mathbb{Q}$ -characters of  $G$ . It contains  $G(\mathbb{Q})$  as a cocompact discrete subgroup, and one has a decomposition

$$G(\mathbb{A}) = A_G(\mathbb{R})^+ \times G(\mathbb{A})^1.$$

Automorphic forms are affiliated with a central character, which we fix throughout. That is, we pick an arbitrary continuous (possibly non-unitary) character

$$\omega : Z_G(\mathbb{Q}) \backslash Z_G(\mathbb{A}) \rightarrow \mathbb{C}^*,$$

and consider the Hilbert space  $L_G^2(\omega)$  of all measurable  $\omega$ -central functions

$$f : G(\mathbb{Q}) \backslash G(\mathbb{A}) \rightarrow \mathbb{C}, \quad \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})^1} |f(x)|^2 dx < \infty.$$

The right regular representation of  $G(\mathbb{A})$  is completely reducible, and  $L_G^2(\omega)$  breaks up into (irreducible) automorphic representations  $\pi = \pi_\infty \otimes \pi_f$ , each occurring with finite multiplicity  $m_G(\pi)$ . The space of automorphic forms  $\mathcal{A}_G(\omega)$ , is the dense subspace of smooth functions  $f$  satisfying the usual finiteness properties under the action of  $K_\infty$ , and the center of the universal enveloping algebra at infinity. We will restrict ourselves to *algebraic*  $\pi$ . That is, we will assume  $\pi_\infty$  is the restriction of an irreducible algebraic (finite-dimensional) representation

$$\xi : G_{\mathbb{C}} \rightarrow \text{GL}(W),$$

which we fix throughout. Its isotypic component is  $\xi \otimes \mathcal{A}_G(\omega)$ , where we let

**Definition 1.**  $\mathcal{A}_{G,\xi}(\omega) = \text{Hom}_{G(\mathbb{R})}(\xi, \mathcal{A}_G(\omega)) = (\xi^\vee \otimes \mathcal{A}_G(\omega))^{G(\mathbb{R})}$ .

This is an admissible smooth representation of  $G(\mathbb{A}_f)$ , which breaks up as a direct sum  $\oplus_{\pi} m_G(\pi) \pi_f$ , summing over automorphic  $\pi$ , of central character  $\omega_{\pi} = \omega$ , such that  $\pi_{\infty} = \xi$ . We view elements of  $\mathcal{A}_{G,\xi}(\omega)$  as vector-valued functions.

**Lemma 1.** *As a  $G(\mathbb{A}_f)$ -module,  $\mathcal{A}_{G,\xi}(\omega)$  can be identified with the space of all  $\omega_f$ -central smooth functions*

$$f : G(\mathbb{A}_f) \rightarrow W^{\vee}, \quad f(\gamma_f x) = \xi^{\vee}(\gamma_{\infty}) f(x), \quad \forall \gamma \in G(\mathbb{Q}).$$

*Proof.* One introduces a third space, consisting of all smooth  $\omega$ -central functions

$$f : G(\mathbb{Q}) \backslash G(\mathbb{A}) \rightarrow W^{\vee}, \quad f(xg) = \xi^{\vee}(g)^{-1} f(x), \quad \forall g \in G(\mathbb{R}).$$

Such a function  $f$  gives a  $G(\mathbb{R})$ -map  $\xi \rightarrow \mathcal{A}_G(\omega)$  by sending a vector  $w \in W$  to the automorphic form  $g \mapsto \langle f(g), w \rangle$ . On the other hand, restriction to  $G(\mathbb{A}_f)$  identifies it with the space of functions in the lemma.  $\square$

*Remark.* We always assume  $\xi$  and  $\omega$  are compatible, that is  $\omega_{\infty} = \xi|_{Z_G(\mathbb{R})}$ .

By smoothness, as  $K$  varies over all compact open subgroups of  $G(\mathbb{A}_f)$ , one has

$$\mathcal{A}_{G,\xi}(\omega) = \varinjlim_K \mathcal{A}_{G,\xi}(\omega)^K,$$

where  $\mathcal{A}_{G,\xi}(\omega)^K$  is the subspace of  $K$ -invariants, a module for the Hecke algebra  $\mathcal{H}_{G,K}$  of all  $K$ -biinvariant compactly supported  $\mathbb{C}$ -valued functions on  $G(\mathbb{A}_f)$ . Again, for this subspace to be nonzero, we need  $K$  and  $\omega$  to be compatible, in the sense that  $\omega_f$  is trivial on  $Z_G(\mathbb{A}_f) \cap K$ .

*Example.* When  $\xi = 1$ , we are just looking at the space  $\mathcal{A}_{G,1}(\omega)$  of all  $\omega_f$ -central smooth  $\mathbb{C}$ -valued functions on the profinite (hence compact) set

$$\tilde{S} = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) = \varinjlim_K S_K, \quad S_K = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K.$$

Moreover,  $\mathcal{A}_{G,1}(\omega)^K$  is the space of  $\omega_f$ -central functions on the finite set  $S_K$ .

## 2.2 The $p$ -adic case

### 2.2.1 Notation

We fix a prime number  $p$ , an algebraic closure  $\bar{\mathbb{Q}}_p$ , together with an (algebraic) isomorphism  $\iota : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_p$ . We will occasionally make use of an algebraic closure  $\bar{\mathbb{Q}}$ , always assumed to be endowed with an embedding  $\iota_{\infty} : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ . Correspondingly,  $\iota_p = \iota \circ \iota_{\infty}$  is an embedding  $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ . Via  $\iota$ , we base change  $\xi$  to an algebraic representation over  $\bar{\mathbb{Q}}_p$ ,

$$\iota \xi : G_{\bar{\mathbb{Q}}_p} \rightarrow \mathrm{GL}(\iota W), \quad \iota W = W \otimes_{\mathbb{C}, \iota} \bar{\mathbb{Q}}_p.$$

Our central character  $\omega$  has a  $p$ -adic avatar, the continuous character

$$\omega_{f,p} : Z_G(\mathbb{Q}) \backslash Z_G(\mathbb{A}_f) \rightarrow \bar{\mathbb{Q}}_p^*, \quad \omega_{f,p}(z) = \iota \omega_{\xi}(z_p) \cdot \iota \omega_f(z).$$

### 2.2.2 Classical $p$ -adic automorphic forms

All constructions of the previous section can be transferred to  $\bar{\mathbb{Q}}_p$  via  $\iota$ . When we put an  $\iota$  in front, we mean tensoring by  $\bar{\mathbb{Q}}_p$ , as in  $\iota W = W \otimes_{\mathbb{C}, \iota} \bar{\mathbb{Q}}_p$ .

**Lemma 2.** *As a  $G(\mathbb{A}_f)$ -module,  $\iota \mathcal{A}_{G, \xi}(\omega)^K$  can be identified with the space of all  $\omega_{f, p}$ -central functions (smooth away from  $p$ )*

$$f : G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \rightarrow \iota W^\vee, \quad f(xk) = \iota \xi^\vee(k_p)^{-1} f(x), \quad \forall k \in K.$$

*Proof.* Given a complex form  $f$ , as in the previous lemma, one associates the function  $x \mapsto \iota \xi^\vee(x_p)^{-1} \iota f(x)$ . It is easy to check that one can recover  $f$ .  $\square$

**Definition 2.**  $\mathcal{C}_G(\omega) = \{\text{continuous } \omega_{f, p}\text{-central } G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \xrightarrow{f} \bar{\mathbb{Q}}_p\}$ .

Any function  $f$ , as in the lemma, yields a  $K$ -map  $\iota \xi \rightarrow \mathcal{C}_G(\omega)$  by sending  $w \in \iota W$  to the continuous (in fact, locally algebraic) function  $g \mapsto \langle f(g), w \rangle$ , and vice versa. Here  $K$  acts on  $\iota \xi$  through the projection to  $G(\mathbb{Q}_p)$ . We have shown,

$$\iota \mathcal{A}_{G, \xi}(\omega)^K = \text{Hom}_K(\iota \xi, \mathcal{C}_G(\omega)) = (\iota \xi^\vee \otimes \mathcal{C}_G(\omega))^K.$$

Note that the image of  $K$  in  $G(\mathbb{Q}_p)$  is compact open, hence Zariski dense, so that  $\iota \xi$  is an irreducible representation of  $K$ . Let us look at its isotypic subspace  $\mathcal{C}_G(\omega)[\iota \xi]$ . That is, the sum of all  $K$ -stable subspaces isomorphic to  $\iota \xi$ . This is a semisimple  $K$ -representation, and  $\text{Hom}_K(\iota \xi, \mathcal{C}_G(\omega))$  is its multiplicity space,

$$\iota \xi \otimes \iota \mathcal{A}_{G, \xi}(\omega)^K \xrightarrow{\sim} \mathcal{C}_G(\omega)[\iota \xi] \subset \mathcal{C}_G(\omega).$$

As  $K$  varies, these identifications are compatible with inclusions among the spaces  $\mathcal{A}_{G, \xi}(\omega)^K$ . Taking the direct limit, we end up with the injection

$$\varinjlim_K \iota \xi \otimes \iota \mathcal{A}_{G, \xi}(\omega)^K \hookrightarrow \mathcal{C}_G(\omega).$$

It can be checked that this map is  $G(\mathbb{A}_f)$ -equivariant. The image is the subspace of locally  $\xi$ -algebraic functions. Altogether, we arrive at our key result:

**Theorem 1.** *There is an injective  $G(\mathbb{A}_f)$ -map  $\iota \xi \otimes \iota \mathcal{A}_{G, \xi}(\omega) \hookrightarrow \mathcal{C}_G(\omega)$ .*

### 2.2.3 Existence of invariant norms

The space  $\mathcal{C}_G(\omega)$ , being a subspace of  $\mathcal{C}(\tilde{S}, \bar{\mathbb{Q}}_p)$ , has a natural sup-norm,

$$\|f\| = \sup_{x \in G(\mathbb{A}_f)} |f(x)|_p = \max_{x \in G(\mathbb{A}_f)} |f(x)|_p,$$

which is obviously invariant under the  $G(\mathbb{A}_f)$ -action, that is  $\|g \cdot f\| = \|f\|$ .

**Corollary 1.** *If  $\pi = \xi \otimes \pi_f$  is an automorphic representation of  $G(\mathbb{A})$ , then  $\iota \xi \otimes \iota \pi_f$  has a natural  $G(\mathbb{A}_f)$ -invariant norm. (Here  $G(\mathbb{A}_f^p)$  acts through  $\iota \pi_f^p$ , and  $G(\mathbb{Q}_p)$  acts diagonally.)*

Since  $\iota \xi \otimes \iota \pi_f = (\iota \xi \otimes \iota \pi_p) \otimes \iota \pi_f^p$ , we deduce:

**Corollary 2.** *If  $\pi_p$  is an irreducible admissible representation of  $G(\mathbb{Q}_p)$ , which extends to an automorphic representation of  $G(\mathbb{A})$  of weight  $\xi$ , then  $\iota \xi \otimes \iota \pi_p$  has a  $G(\mathbb{Q}_p)$ -invariant norm.*

This norm is far from canonical. There may be many ways to extend  $\pi_p$ .

### 3 A Grunwald-Wang type theorem

#### 3.1 The Grunwald-Wang theorem for $GL(1)$

We briefly recall, from [AT, p. 103], the following result of Grunwald (as corrected by Wang).

**Theorem 2.** *Given a number field  $F$ , a finite set of places  $S$ , and for each  $v \in S$  a character  $\chi_v$  of  $F_v^*$  of finite order, there exists a finite order Hecke character  $\chi$  of  $F$  extending  $\chi_S = \otimes_{v \in S} \chi_v$ .*

Furthermore, the order of  $\chi$  can be taken to be the least common multiple of the orders of the  $\chi_v$ , unless a special case occurs (where the order of  $\chi$  becomes twice that). Given an arbitrary  $\chi_S$ , we see that it can be extended to a Hecke character conditionally: Precisely when some twist  $\chi_S | \cdot |_S^s$  is of finite order. This is a constraint among the  $\{\chi_v\}_{v \in S}$  (as  $s \in \mathbb{C}$  depends only on  $S$ ).

#### 3.2 Clozel's argument on limit multiplicities

We will use the trace formula in its absolute simplest form. Namely, we will assume, for a moment, that  $G$  is *semisimple*. We keep all other assumptions. In particular,  $G(\mathbb{R})$  is compact. The trace formula for  $G$  is the following identity,

$$\mathrm{tr}(\phi : L_G^2) = \sum_{\pi} m_G(\pi) \mathrm{tr} \pi(\phi) = \sum_{\{\gamma\}} \mathrm{vol}(G_{\gamma}(\mathbb{Q}) \backslash G_{\gamma}(\mathbb{A})) O_{\gamma}(\phi),$$

valid for any test function  $\phi \in \mathcal{C}_c^{\infty}(G(\mathbb{A}))$ . On the spectral side, we are summing over all automorphic representations  $\pi$ . On the geometric side, the sum ranges over  $\gamma \in G(\mathbb{Q})$ , up to conjugacy. We denote by  $G_{\gamma}$  its stabilizer, and by  $O_{\gamma}$  the orbital integral. Measures are chosen compatibly.

We wish to quickly outline an argument of Clozel, giving an analogue of the Grunwald-Wang theorem for  $G$ . We start off with a finite set of places  $S$  of  $\mathbb{Q}$ , which we assume contains  $\infty$ . At each  $v \in S$ , we are given a discrete series representation  $\pi_v^{\circ}$  of  $G(\mathbb{Q}_v)$  (that is, its matrix coefficients are square-integrable).

**Theorem 3.** *There is a function  $\phi_v^{\circ} \in \mathcal{C}_c^{\infty}(G(\mathbb{Q}_v))$  such that, for every tempered irreducible admissible representation  $\pi_v$ ,*

$$\mathrm{tr} \pi_v(\phi_v^{\circ}) = \begin{cases} 1, & \pi_v = \pi_v^{\circ} \\ 0, & \pi_v \neq \pi_v^{\circ} \end{cases}$$

(Such a  $\phi_v^{\circ}$  is called a *pseudo-coefficient* of  $\pi_v^{\circ}$ .)

*Proof.* For  $v = \infty$  this is in [CD]. The case  $v \neq \infty$  is in [C, p. 278].  $\square$

*Note.* There may be non-tempered  $\pi_v$ , for which  $\mathrm{tr} \pi_v(\phi_v^{\circ}) \neq 0$ , but only finitely many. See [C, p. 269] and [C, p. 280]. Let us introduce  $\phi_S^{\circ} = \otimes_{v \in S} \phi_v^{\circ}$ . Then  $\mathrm{tr} \pi_S(\phi_S^{\circ}) \neq 0$  for only finitely many representations  $\pi_S^{\circ} = \pi_{S,0}, \dots, \pi_{S,r}$ .

With this choice of  $\phi_S^{\circ}$ , the spectral side becomes

$$\sum_{\pi^S} m_G(\pi_S^{\circ} \otimes \pi^S) \mathrm{tr} \pi^S(\phi^S) + \sum_{i=1}^r \sum_{\pi^S} m_G(\pi_{S,i} \otimes \pi^S) \mathrm{tr} \pi_{S,i}(\phi_S^{\circ}) \mathrm{tr} \pi^S(\phi^S)$$



for all  $\phi^S \in \mathcal{C}_c^\infty(G(\mathbb{A}^S))$ . We will take this  $\phi^S$  to be of the following form:

$$\phi^S = \text{vol}(K^S)^{-1} \cdot \text{char}_{K^S},$$

where  $K^S \subset G(\mathbb{A}^S)$  is a compact open subgroup, to be varied. With this choice, the spectral side turns into

$$\dim \text{Hom}_{G(\mathbb{Q}_S)}(\pi_S^\circ, (L_G^2)^{K^S}) + \sum_{i=1}^r \dim \text{Hom}_{G(\mathbb{Q}_S)}(\pi_{S,i}, (L_G^2)^{K^S}) \text{tr} \pi_{S,i}(\phi_S^\circ)$$

In some sense, the key ingredient of Clozel's proof is the following limit multiplicity formula, based on a method of DeGeorge-Wallach.

**Lemma 3.**  $\lim_{K^S \rightarrow 1} \text{vol}(K^S) \dim \text{Hom}_{G(\mathbb{Q}_S)}(\pi_{S,i}, (L_G^2)^{K^S}) = 0$  for  $i > 0$ .

*Proof.* This is (a weak version of) Lemma 8, [C, p. 274].  $\square$

Now, let us focus on the geometric side,

$$\sum_{\{\gamma\}} \text{vol}(G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})) O_{\gamma_S}(\phi_S^\circ) O_{\gamma^S}(\phi^S).$$

Here, by Lemma 5 in [C, p. 271], for sufficiently small  $K^S$ , the factor  $O_{\gamma^S}(\phi^S) = 0$  unless  $\gamma$  is unipotent. Since  $G$  is  $\mathbb{Q}$ -anisotropic, this means  $\gamma = 1$ . In the limit, as  $K^S \rightarrow 1$ , the geometric side reduces to just one term,

$$\text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})) \phi_S^\circ(1) \text{vol}(K^S)^{-1}.$$

Here  $\phi_S^\circ(1) = d(\pi_S^\circ) > 0$  is the formal degree, by the Plancherel formula. See Lemma 9 and 12 in [C]. Putting all this together, we arrive at the following limit formula,

**Theorem 4.**  $\text{vol}(K^S) \dim \text{Hom}_{G(\mathbb{Q}_S)}(\pi_S^\circ, (L_G^2)^{K^S}) \xrightarrow{K^S \rightarrow 1} \text{vol}(G(\mathbb{Q}) \backslash G(\mathbb{A})) d(\pi_S^\circ).$

This is a weak version of Theorems 1A and 1B in [C], which control ramification away from just one prime. We will not need this. On the other hand, Clozel's theorems give lower bounds for  $\liminf_{K^S \rightarrow 1}$ , not exact limits.

What will be crucial for the applications we have in mind later on, is the following extension theorem, in the vein of Grunwald-Wang,

**Corollary 3.** *Let  $G$  be a semisimple anisotropic  $\mathbb{Q}$ -group. Given a discrete series representation  $\pi_S^\circ$  of  $G(\mathbb{Q}_S)$ , where  $S$  is a finite set of places of  $\mathbb{Q}$ , there is an automorphic representation  $\pi$  of  $G(\mathbb{A})$  such that  $\pi_S = \pi_S^\circ$ .*

## 4 Invariant norms on discrete series

### 4.1 Forms of algebraic groups

We will quote (and use) a result of Borel and Harder on locally prescribed forms of algebraic groups. Recall, if  $G$  is an algebraic group over a field  $F$ , an  $F$ -form of  $G$  is an  $F$ -group  $G'$  isomorphic to  $G$  over the algebraic closure  $\bar{F}$ . This gives

rise to a cocycle  $c : \text{Gal}(\bar{F}/F) \rightarrow \text{Aut}(G)$  in the obvious way, and identifies the set of equivalence classes of forms with the non-abelian Galois cohomology set,

$$H^1(F, \text{Aut}(G)).$$

We will take  $F$  to be a number field. For each place  $v$  of  $F$ , there is an obvious restriction map

$$H^1(F, \text{Aut}(G)) \rightarrow H^1(F_v, \text{Aut}(G)),$$

which on forms corresponds to extending scalars  $G' \rightsquigarrow G'_v = G' \otimes_F F_v$ .

**Theorem 5.** *Let  $F$  be a number field,  $S$  a finite set of places of  $F$ , and  $G$  an (absolutely) almost simple  $F$ -group which is either simply connected or of adjoint type. Then the canonical restriction map is surjective,*

$$H^1(F, \text{Aut}(G)) \twoheadrightarrow \prod_{v \in S} H^1(F_v, \text{Aut}(G)).$$

*In other words, given an  $F_v$ -form  $G'_v$  for each  $v \in S$ , there is an  $F$ -form  $G'$  equivalent to  $G'_v$  at places in  $S$ .*

*Proof.* This is Theorem B in [BH].  $\square$

If  $v$  is a real (infinite) place of  $F$ , there is always a unique compact form  $G'_v$ , up to equivalence. The corresponding cocycle  $c$  is essentially given by the Cartan involution. We immediately deduce the following existence result, which will be used in the next section.

**Corollary 4.** *Let  $G$  be an almost simple  $\mathbb{Q}_p$ -group which is either simply connected or of adjoint type. Then there is a model over  $\mathbb{Q}$ , still denoted by  $G$ , such that  $G(\mathbb{R})$  is compact.*

*Proof.* The group  $G_{\bar{\mathbb{Q}}_p} \simeq G_{\mathbb{C}}$  has a split model over  $\mathbb{Q}$  (even over  $\mathbb{Z}$ , this is the theory of Chevalley groups), which we will denote by  $G^*$ . We apply the Theorem to this group, with  $S = \{\infty, p\}$ . At  $\infty$  we take the compact form of  $G_{\mathbb{R}}^*$ , at  $p$  we take  $G$ .  $\square$

## 4.2 The simple case

The following result is at the heart of our method.

**Lemma 4.** *Let  $G$  be an almost simple  $\mathbb{Q}_p$ -group which is either simply connected or of adjoint type. Let  $\xi$  be any irreducible algebraic representation of  $G_{\bar{\mathbb{Q}}_p}$ , and  $\pi$  be any discrete series representation of  $G(\mathbb{Q}_p)$  (both over  $\bar{\mathbb{Q}}_p$ ). Then the locally algebraic representation  $\xi \otimes \pi$  carries a norm, which is invariant under the  $G(\mathbb{Q}_p)$ -action.*

*Proof.* The key is to embed this in a global situation. Thus, as in the previous Corollary, we first find a  $\mathbb{Q}$ -model  $G$  such that  $G(\mathbb{R})$  is compact. With a choice of an isomorphism  $\iota : \mathbb{C} \rightarrow \bar{\mathbb{Q}}_p$ , we can confuse  $\xi$  and  $\pi$  with representations over  $\mathbb{C}$  (of  $G_{\mathbb{C}}$  and  $G(\mathbb{Q}_p)$  respectively). We will change notation, and denote the previous  $\pi$  by  $\pi_p^\circ$ . Also, we let  $\pi_\infty^\circ = \xi|_{G(\mathbb{R})}$ . Both are discrete series, so by Corollary 3 there is an automorphic representation  $\pi$  of  $G(\mathbb{A})$  such that  $\pi_\infty = \xi$  and  $\pi_p = \pi_p^\circ$ . By Corollary 2, we see that  $\iota\xi \otimes \iota\pi_p^\circ$  has an invariant norm.  $\square$

### 4.3 The semisimple case

From the simple case, we derive the semisimple case,

**Lemma 5.** *Let  $G$  be a connected semisimple  $\mathbb{Q}_p$ -group. Let  $\xi$  be any irreducible algebraic representation of  $G_{\bar{\mathbb{Q}}_p}$ , and  $\pi$  be any discrete series representation of  $G(\mathbb{Q}_p)$  (both over  $\bar{\mathbb{Q}}_p$ ). Then the locally algebraic representation  $\xi \otimes \pi$  carries a norm, which is invariant under the  $G(\mathbb{Q}_p)$ -action.*

*Proof.* Now, suppose  $G$  is any connected semisimple  $\mathbb{Q}_p$ -group, and let  $G^{\text{sc}} \twoheadrightarrow G$  be its universal covering over  $\mathbb{Q}_p$ , see [PR]. The kernel  $\pi_1(G)$  is finite. Being simply connected,  $G^{\text{sc}}$  is an actual direct product  $G_1 \times \cdots \times G_r$ , of finitely many simply connected simple groups  $G_i$ . By the main theorem of [Si], the restriction of  $\pi$  to  $G^{\text{sc}}$  is a direct sum of finitely many irreducible admissible representations,

$$\pi|_{G^{\text{sc}}} \simeq \bigoplus_{j=1}^s (\tau_{1,j} \otimes \cdots \otimes \tau_{r,j}),$$

where  $\tau_{i,j}$  is a discrete series representation of  $G_i(\mathbb{Q}_p)$ . The restriction  $\xi|_{G^{\text{sc}}}$  remains irreducible, and we continue to denote it simply by  $\xi$ . It factors as a tensor product  $\xi_1 \otimes \cdots \otimes \xi_r$ , where  $\xi_i$  is an irreducible algebraic representation of  $G_{i,\bar{\mathbb{Q}}_p}$ . According to Lemma 4, each  $\xi_i \otimes \tau_{i,j}$  has a norm  $\|\cdot\|_{i,j}$ , invariant under the action of  $G_i(\mathbb{Q}_p)$ . On the tensor product, where  $j$  is fixed for now,

$$(\xi_1 \otimes \tau_{1,j}) \otimes \cdots \otimes (\xi_r \otimes \tau_{r,j}),$$

we put the tensor product norm, see [Sc, p. 110] and Proposition 17.4 therein. It has the property that

$$\|v_1 \otimes \cdots \otimes v_r\|_j = \|v_1\|_{1,j} \cdots \|v_r\|_{r,j},$$

with  $v_i \in \xi_i \otimes \tau_{i,j}$ . It is defined, for sums of pure tensors, by the formula

$$\|v\|_j = \inf \{ \max \|v_1\|_{1,j} \cdots \|v_r\|_{r,j} : v = \sum v_1 \otimes \cdots \otimes v_r \}.$$

Here the maximum is over the same index set as the summation. The infimum is over all possible expressions for  $v$ . This tensor product norm  $\|\cdot\|_j$  is clearly invariant under  $G^{\text{sc}}(\mathbb{Q}_p)$ . Taking the maximum of all these, over  $j = 1, \dots, s$ , we have constructed a  $G^{\text{sc}}(\mathbb{Q}_p)$ -invariant norm  $\|\cdot\|$  on  $\xi \otimes \pi$ . Now, to make it invariant under  $G(\mathbb{Q}_p)$ , we note that

$$G(\mathbb{Q}_p)/\text{im}(G^{\text{sc}}(\mathbb{Q}_p) \rightarrow G(\mathbb{Q}_p)) \subset H^1(\mathbb{Q}_p, \pi_1(G))$$

is a finite abelian group. Pick a set of representatives  $R$ , and replace  $\|\cdot\|$  with

$$\|v\|' = \max_{g \in R} \|g \cdot v\|.$$

By construction, this modification  $\|\cdot\|'$  is a  $G(\mathbb{Q}_p)$ -invariant norm on  $\xi \otimes \pi$ .  $\square$

### 4.4 The reductive case

From the semisimple case, we derive the general reductive case.

**Definition 3.** An irreducible admissible complex representation  $\pi$  of  $G(\mathbb{Q}_p)$  is essentially discrete series if a twist  $\pi \otimes \nu$  is (unitary) discrete series, for some smooth character  $\nu : G(\mathbb{Q}_p) \rightarrow \mathbb{C}^*$ . The essentially discrete series representations over  $\mathbb{Q}_p$  are those of the form  $\iota\pi$ , for some isomorphism  $\iota : \mathbb{C} \rightarrow \mathbb{Q}_p$ .

*Remark.* To put this definition (over  $\mathbb{Q}_p$ ) on more solid ground, we would like to know that we can in fact pick any  $\iota$ . In other words, whether any  $\text{Aut}(\mathbb{C})$ -conjugate of an essentially discrete series representation is again essentially discrete series<sup>3</sup>. This is predicted by the local Langlands conjecture (the parameter does not map into a proper Levi). If  $\sigma \in \text{Aut}(\mathbb{C})$ , the matrix coefficients of  $\sigma\pi$  are  $\sigma$ -conjugates of matrix coefficients of  $\pi$ . Hence, it is certainly true for supercuspidals, but square integrability seems to be a problem. We should mention that at least it is known to be true for  $\text{GL}(n)$ . Indeed the work of Bernstein-Zelevinsky shows that the essentially discrete series representations for  $\text{GL}(n)$  coincides with the generalized Steinberg representations  $Q(\Delta)$ , built from a segment  $\Delta$  of supercuspidals, and  $\sigma Q(\Delta) = Q(\sigma\Delta)$  in a suitable (rational) normalization. See [Ku] for a nice exposition of the Langlands classification.

**Theorem 6.** Let  $G$  be a connected reductive group over  $\mathbb{Q}_p$ . Let  $\xi$  be any irreducible algebraic representation of  $G_{\mathbb{Q}_p}$ , and  $\pi$  be any essentially discrete series representation of  $G(\mathbb{Q}_p)$  (both over  $\mathbb{Q}_p$ ). Then the locally algebraic representation  $\xi \otimes \pi$  admits a  $G(\mathbb{Q}_p)$ -invariant norm if and only if its central character  $\omega_\xi \cdot \omega_\pi$  is integral (that is, maps into  $\bar{\mathbb{Z}}_p^\times$ ).

*Proof.* The only if part is obvious. We assume  $\omega_\xi \cdot \omega_\pi$  is integral, and seek a norm. The derived subgroup  $G^{\text{der}}$  is semisimple,  $Z_G \cap G^{\text{der}}$  is finite, and

$$1 \rightarrow Z_G \cap G^{\text{der}} \rightarrow Z_G \times G^{\text{der}} \rightarrow G \rightarrow 1.$$

is exact. Here  $Z_G$  is the full identity component of the center. The restriction  $\xi|_{G^{\text{der}}}$  hence remains irreducible, and we will just write  $\xi$ . On the other hand, the restriction  $\pi|_{G^{\text{der}}(\mathbb{Q}_p)}$  may not be, but it breaks up as a direct sum

$$\pi|_{G^{\text{der}}(\mathbb{Q}_p)} \simeq \tau_1 \oplus \cdots \oplus \tau_r$$

of discrete series representations  $\tau_i$  of  $G^{\text{der}}(\mathbb{Q}_p)$ . For example, see [Ta, p. 381] and [Ta, p. 385]. By Lemma 5, there is a norm  $\|\cdot\|_i$  on  $\xi \otimes \tau_i$ , invariant under  $G^{\text{der}}(\mathbb{Q}_p)$ . Their maximum defines a  $G^{\text{der}}(\mathbb{Q}_p)$ -invariant norm  $\|\cdot\|$  on  $\xi \otimes \pi$ , which is automatically  $Z_G(\mathbb{Q}_p)$ -invariant, by our assumption on the central character.

$$G(\mathbb{Q}_p)/Z_G(\mathbb{Q}_p)G^{\text{der}}(\mathbb{Q}_p) \subset H^1(\mathbb{Q}_p, Z_G \cap G^{\text{der}})$$

is a finite abelian group. Pick representatives  $R$ , and replace  $\|\cdot\|$  with

$$\|v\|' = \max_{g \in R} \|g \cdot v\|.$$

This is independent of  $R$ , and defines a  $G(\mathbb{Q}_p)$ -invariant norm on  $\xi \otimes \pi$ .  $\square$

Taking  $G = \text{Res}_{F/\mathbb{Q}_p} \text{GL}(n)$ , for a finite extension  $F/\mathbb{Q}_p$ , yields:

**Corollary 5.** Conjecture 5.5 in [BS] holds true.

*Proof.* As already mentioned, by Bernstein-Zelevinsky, the essentially discrete series representations of  $\text{GL}(n)$  are precisely the generalized Steinberg representations.  $\square$

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<sup>3</sup>Marko Tadic informs me that, at least for classical groups, this is known for generic representations.

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